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Robust Positioning of Service Units

J. Puerto^{1,*} and A. M. Rodríguez-Chía^{2,*}

¹Dept. Estadística e Investigación Operativa, Universidad de Sevilla,
Fac. de Matemáticas, Sevilla, Spain

²Dept. Estadística e Investigación Operativa, Universidad de Cádiz, Fac. de Ciencias
del Mar, Polígono Río San Pedro, Cádiz, Spain

ABSTRACT

In this paper, we address the problem of locating mobile service units to cover random incidents. The model does not assume complete knowledge of the probability distribution of the location of the incident to be covered. Instead, only the mean value of that distribution is known. We propose the minimization of the maximum expected response time as an effectiveness measure for the model. Thus, the solution obtained is robust with respect to any probability distribution. The cases of one and two service units under the nearest allocation rule are studied in the paper. For both problems, the optimal solutions are shown to be degenerate distributions for the servers.

Key Words: Location theory; Stochastic problems.

1. INTRODUCTION

In distribution systems and in continuous location models, a common problem is to find the optimal placement of one or more servers minimizing the distances to a given set of points. All the models considered so far in the literature assume that the positions of

*Correspondence: J. Puerto, Dept. Estadística e Investigación Operativa, Universidad de Sevilla, Fac. de Matemáticas, C/Tarfia s/n, 41012 Sevilla, Spain; E-mail: puerto@cica.es. A. M. Rodríguez-Chía, Dept. Estadística e Investigación Operativa, Universidad de Cádiz, Fac. de Ciencias del Mar, Polígono Río San Pedro, 11510 Puerto Real, Cádiz, Spain; E-mail: antonio.rodriguezchia@uca.es.

47 these points are either deterministic or distributed according to a known probability
 48 distribution on the family of Borel sets in \mathbb{R}^n (see for instance Anderson and Fontenot,^[1]
 49 Carrizosa, Muñoz-Márquez and Puerto,^[2] Larson and Odoni,^[5] Levine,^[6] De Palma, Liu
 50 and Thisse,^[7] among others).

51 However, it is easy to find situations in the real-world where the hypothesis of
 52 complete knowledge of this probability distribution is unrealistic. In this paper, we
 53 propose a more general model where only the mean value of this distribution is known.
 54 This assumption is not really restrictive because we can obtain good estimates of the
 55 unknown mean value by sampling. Although more information can be obtained from the
 56 sample, our model only needs the estimation of the mean value, which is a very well-
 57 solved problem in mathematical statistics. A real-world application of such models is, for
 58 instance, the problem of locating a read/write head of a computer hard-disk to easily
 59 access the stored data. Similarly, our framework includes the problem of positioning
 60 police-cars that must cover incidents where the law is being broken, and positioning idle
 61 elevators to minimize response time (see Vickson, Gerchak and Rotem^[10] or Smith^[9] for a
 62 different analyses assuming that the distribution of the data is known). Indeed, in these
 63 cases, usually the distribution of the places where the law will be broken, the data are
 64 stored, or the elevator is needed is not known. Nevertheless, it would be less restrictive to
 65 assume that either we know the mean value for these distributions or we may estimate it by
 66 means of an empirical study.

67 When the probability distribution of the position of the incident is unknown, the
 68 classical minimization of the expected distances is not possible. Therefore, alternative
 69 approaches have to be considered. In this paper, we propose a robust alternative consisting
 70 of minimizing the maximum expected distance within the whole family of probability
 71 measures which model the incident (see Gallego^[4] and Puerto and Fernández^[8] for similar
 72 analyses applied to different problems in Operations Research).

73 Let $\mathcal{F}(\lambda)$ and $\mathcal{G}(\mu)$ be the families of random variables (r.v.) (given by their
 74 cumulative distribution functions (c.d.f.) defined on the n -dimensional hypercube $[0,1]^n$
 75 with mean values $\lambda \in \mathbb{R}^n$ for $\mathcal{F}(\lambda)$ and $\mu \in \mathbb{R}^n$ for $\mathcal{G}(\mu)$, that is,

$$76 \quad \mathcal{F}(\lambda) = \{X : \text{r.v. on } [0, 1]^n \text{ with c.d.f. } F_X, \int_{[0,1]^n} x \, dF_X(x) = \lambda\},$$

$$77 \quad \mathcal{G}(\mu) = \{A : \text{r.v. on } [0, 1]^n \text{ with c.d.f. } G_A, \int_{[0,1]^n} a \, dG_A(a) = \mu\}.$$

81 Define

$$82 \quad \mathcal{F} := \bigcup_{\lambda \in [0,1]^n} \mathcal{F}(\lambda).$$

83
 84 The families \mathcal{F} and $\mathcal{G}(\mu)$ are the sets of random variables which model the position of
 85 the server and the incident, respectively. It is worth noting that we have defined these
 86 random variables in the n -dimensional hypercube $[0,1]^n$, but they can be extended to any
 87 hyperrectangle by a linear transformation.

88 As previously mentioned, some authors have studied the problem of minimizing the
 89 expected distance to the random incident, i.e.,

$$90 \quad \min_{X \in \mathcal{F}} \int_{[0,1]^n} \int_{[0,1]^n} d(x, a) \, dG_A(a) \, dF_X(x),$$

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93 where d is a measure of distance, X is a r.v. with c.d.f. F_X , representing the position of the
 94 server, and A is a r.v. with c.d.f. G_A , representing the position of the incident.

95 Our model does not assume any *a priori* knowledge about the probability distribution
 96 of the incident apart from its mean value. That is to say, we have almost complete
 97 uncertainty about where the incident will take place, and we search for the policy that an
 98 emergency unit, X , has to follow to minimize the maximum expected distance to any
 99 random incident. Therefore, the problem is

$$100 \quad \min_{X \in \mathcal{F}} \max_{A \in \mathcal{G}(\mu)} \int_{[0,1]^n} \int_{[0,1]^n} \|x - a\|_1 dG_A(a) dF_X(x), \quad (1)$$

103 where $\|\cdot\|_1$ is the l_1 -norm in \mathbb{R}^n , so that, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have that

$$104 \quad \|x\|_1 = \sum_{i=1}^n |x_i|.$$

107 The readers should note that there are essentially two kinds of factors that influence
 108 the formulation of the problem: 1) the dimension n of the space where the incidents occur;
 109 and 2) the number of service units to be located.

111 It is also worth noting that this problem formulation can be used to model the above
 112 mentioned real-world situations because: a) we do not need to know the distribution of the
 113 incident; and b) the read/write head only admits displacements following the directions of
 114 the coordinate axes; and both the highway and the trajectory of the elevator can be
 115 considered like line segments where displacements are linear. Thus, the l_1 -norm is an
 116 appropriate measure of distance.

117 Finally, the formulation (1) gives us a new interpretation of the solutions obtained in
 118 terms of statistics. As we shall show in the paper, the optimal probability distributions for
 119 our problem are degenerate random variables. Since principal points of probability
 120 distributions are those points optimizing some effectiveness measure (see Flury^[3]), we can
 121 see our solutions as a generalization of the principal points, but now we are optimizing
 122 over a family of distributions with a fixed mean rather than the values of a single
 123 probability distribution.

124 The paper is organized as follows. In Section 2, we consider the problem of locating a
 125 single facility; we first study the problem considering the service unit as a degenerate
 126 random variable and then we extend these results to the general case with any random
 127 variable. In Section 3, we consider the two-facility problem under the nearest allocation
 128 rule and we follow a scheme similar to that followed in Section 2. In Section 4, we include
 129 some concluding remarks and possible extensions to the considered model. Finally, in the
 130 Appendix, we include, for the sake of readability, several technical results that have been
 131 used in the paper.

2. THE SINGLE FACILITY PROBLEM

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136 We begin this section by considering the one-dimensional case, then we proceed to
 137 the n -dimensional single facility problem. Let $\mathcal{F}_1(\lambda)$ and $\mathcal{G}_1(\mu)$ be, respectively, the
 138 families of random variables $\mathcal{F}(\lambda)$ and $\mathcal{G}(\mu)$ in the 1-dimensional case. For ease of

139 understanding, we distinguish two cases. In the first case, the server is not allowed to
 140 patrol, i.e., we model the location of the server with a degenerate random variable. In the
 141 second case, the server is allowed to patrol, which means that it is any random variable in
 142 \mathcal{F}_1 . For the first case, the mathematical formulation of the problem is:

$$143 \min_{x \in [0,1]} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]} |x - a| dG_A(a). \quad (2)$$

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 147 **Theorem 2.1** *The optimal positioning policy in the hypothesis of Problem (2) is*

$$148 x^* = \begin{cases} 0 & \text{if } \mu = 0.5 \\ y \text{ for any } y \in [0, 1] & \text{if } \mu = 0.5 \\ 1 & \text{if } \mu > 0.5. \end{cases} \quad (3)$$

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 154 *Remark 2.1* This result states that the optimal location for a fixed service unit when only
 155 the mean value μ of the distribution of the incident is known, is on an extreme point of the
 156 interval of feasible locations for the incident. Further, when $\mu = 0.5$ any point on the
 157 interval is an optimal location of the server.
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 160 **Proof:** By Lemma A.2 in the Appendix we have that

$$161 \min_{x \in [0,1]} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]} |x - a| dG_A(a) = \min_{x \in [0,1]} \max_{A \in \mathcal{G}_1(\mu)} \left(2 \int_0^x G_A(a) da - x + \mu \right). \quad (4)$$

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 163 Hence, we prove that the maximum in the last expression is reached at the random variable
 164 A^* with the following c.d.f.

$$165 G_{A^*}(a) = \begin{cases} 0 & \text{if } a < 0 \\ 1 - \mu & \text{if } 0 \leq a < 1 \\ 1 & \text{if } a \geq 1. \end{cases}$$

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 171 Indeed, since x and μ are constants for the inner maximum in the right hand side of (4), we
 172 have to prove the following inequality

$$173 \int_0^x G_A(a) da - (1 - \mu)x \leq 0, \quad \forall x \in [0, 1], \quad \forall A \in \mathcal{G}_1(\mu). \quad (5)$$

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 177 But, since $\int_0^1 G_A(a) da = 1 - \mu$ (Lemma A.2) and $G_A(\cdot)$ is a distribution function, we are
 178 under the hypotheses of Lemma A.1 which proves the inequality (5).

179 Therefore, the minimization Problem (2) reduces to the following problem:

$$180 \min_{x \in [0,1]} x(1 - 2\mu) + \mu.$$

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 182 Hence, depending on the relative values of μ , we obtain that the optimal positioning x^*
 183 satisfies equation (3). \square
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185 In the second case, the service unit is also allowed to patrol. Initially, we permit the
 186 service unit to be distributed on the interval according to a random variable with the only
 187 condition that its mean value is fixed to λ . Then, we solve the case when λ is not fixed. For
 188 the first case, the problem is

$$189 \min_{X \in \mathcal{F}_1(\lambda)} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]} \int_{[0,1]} |x - a| dG_A(a) dF_X(x). \quad (6)$$

192 **Theorem 2.2** Any random variable $X \in \mathcal{F}_1(\lambda)$ constitutes an optimal policy for
 193 Problem (6).
 194

195 **Proof:** By Lemma A.4, we have that

$$196 \int_{[0,1]} \int_{[0,1]} |x - a| dG_A(a) dF_X(x) = 2 \left(1 - \int_0^1 G_A(y) F_X(y) dy \right) - \lambda - \mu.$$

197 Therefore, using that λ and μ are fixed, we can solve Problem (6) by solving the equivalent
 198 problem

$$199 \max_{X \in \mathcal{F}_1(\lambda)} \min_{A \in \mathcal{G}_1(\mu)} \int_0^1 G_A(y) F_X(y) dy.$$

200 In order to do this, we are going to prove that the inner minimum is achieved by the
 201 random variable A^* such that $P(A^* = 0) = 1 - \mu$ and $P(A^* = 1) = \mu$.

$$202 I_{A,X} := \int_0^1 F_X(y) G_A(y) dy - \int_0^1 F_X(y) (1 - \mu) dy \geq 0.$$

203 Considering $t_0 := t_0(A) \in (0, 1)$ such that $t_0 = \inf\{t \in \mathbb{R} : G_A(t) \geq 1 - \mu\}$ we have
 204 the following inequalities:

$$\begin{aligned} 205 I_{A,X} &= \int_0^{t_0} F_X(y) (G_A(y) - (1 - \mu)) dy + \int_{t_0}^1 F_X(y) (G_A(y) - (1 - \mu)) dy \\ 206 &\geq \int_0^{t_0} F_X(t_0) (G_A(y) - (1 - \mu)) dy + \int_{t_0}^1 F_X(t_0) (G_A(y) - (1 - \mu)) dy \\ 207 &= F_X(t_0) \left(\int_0^1 (G_A(y) - (1 - \mu)) dy \right) = 0, \end{aligned}$$

208 by Lemma A.2. Similarly, Lemma A.2 implies that

$$\begin{aligned} 209 \min_{A \in \mathcal{G}_1(\mu)} \int_0^1 G_A(y) F_X(y) dy + \sum_{a \in (0,1)} a P[A = a] P[X = a] &= \int_0^1 F_X(y) (1 - \mu) dy \\ 210 &= (1 - \lambda)(1 - \mu), \end{aligned}$$

211 regardless of the choice of $X \in \mathcal{F}_1(\lambda)$, and the result follows. \square

Let us consider in the following that no assumptions are made on the mean value, λ , of the random variable modelling the service unit. In this situation, the problem is

$$\min_{X \in \mathcal{F}_1} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]} \int_{[0,1]} |x - a| dG_A(a) dF_X(x). \quad (7)$$

Corollary 2.1 *An optimal positioning policy of Problem (7) is the random variable X^* such that $P[X^* = x^*] = 1$ where x^* was defined in (3).*

Proof: Note that

$$\begin{aligned} & \min_{X \in \mathcal{F}_1} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]} \int_{[0,1]} |x - a| dG_A(a) dF_X(x) \\ &= \min_{\lambda \in [0,1]} \min_{X \in \mathcal{F}_1(\lambda)} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]} \int_{[0,1]} |x - a| dG_A(a) dF_X(x). \end{aligned}$$

Let $H(\lambda) = \min_{X \in \mathcal{F}_1(\lambda)} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]} \int_{[0,1]} |x - a| dG_A(a) dF_X(x)$.

For each $\lambda \in [0, 1]$, by the proof of Theorem 2.2 we have that

$$H(\lambda) = 2(1 - (1 - \lambda)(1 - \mu)) - \lambda - \mu = (1 - 2\mu)\lambda + \mu.$$

Thus, if we look for the minimum in λ we obtain

$$\arg \min_{\lambda \in [0,1]} H(\lambda) = \begin{cases} \{0\} & \text{if } \mu < 0.5 \\ y \text{ for any } y \in [0, 1] & \text{if } \mu = 0.5 \\ \{1\} & \text{if } \mu > 0.5, \end{cases}$$

and the result follows. \square

This corollary shows that it is optimal to park the service unit when no hypotheses are made on the distribution of the service unit and only the mean value of the incident is known. Thus, although patrolling may be good for other reasons such as crime prevention, etc., it is not necessary in order to minimize the maximum expected distance to any random incident.

Finally, we also solve the n -dimensional problem. Indeed, let us consider the problem:

$$\min_{X \in \mathcal{F}} \max_{A \in \mathcal{G}(\mu)} \int_{[0,1]^n} \int_{[0,1]^n} \sum_{i=1}^n |x_i - a_i| dG_A(a) dF_X(x), \quad (8)$$

where $\mu = (\mu_1, \dots, \mu_n)$, $x = (x_1, \dots, x_n)$ and $a = (a_1, \dots, a_n)$. Problem (8) can be written equivalently as follows:

$$\begin{aligned} & \min_{X \in \mathcal{F}} \max_{A \in \mathcal{G}(\mu)} \sum_{i=1}^n \int_{[0,1]} \int_{[0,1]} |x_i - a_i| dG_{A_i}(a_i) dF_{X_i}(x_i) \\ & \leq \min_{X \in \mathcal{F}} \sum_{i=1}^n \max_{A_i \in \mathcal{G}_i(\mu_i)} \int_{[0,1]} \int_{[0,1]} |x_i - a_i| dG_{A_i}(a_i) dF_{X_i}(x_i), \end{aligned} \quad (9)$$

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277 where G_{A_i} and F_{X_i} are the marginal distributions of G_A and F_X respectively.

278 Let A_1^*, \dots, A_n^* be the 1-dimensional random variables attaining the inner maxima and
 279 $G_{A_1^*}, \dots, G_{A_n^*}$ their respective cumulative distribution functions. Consider $dG_{A^*} =$
 280 $dG_{A_1^*} \times \dots \times dG_{A_n^*}$, the measure in the product space generated by the measures
 281 $dG_{A_1^*}, \dots, dG_{A_n^*}$, and let A^* be a n -dimensional random variable with cumulative
 282 distribution G_{A^*} . That means, A^* is a random vector whose components are independent
 283 random variables. Since A^* is feasible for the former maximum in (8), we have that (9)
 284 holds with equality. By a similar argument, we get

$$285$$

$$286 \min_{X \in \mathcal{F}} \max_{A \in \mathcal{G}(\mu)} \int_{[0,1]^n} \int_{[0,1]^n} \sum_{i=1}^n |x_i - a_i| dG_A(a) dF_X(x)$$

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$$289 = \sum_{i=1}^n \min_{X_i \in \mathcal{F}_1} \max_{A_i \in \mathcal{G}_1(\mu_i)} \int_{[0,1]} \int_{[0,1]} |x_i - a_i| dG_{A_i}(a_i) dF_{X_i}(x_i).$$

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292 Thus, we have obtained that the n -dimensional problem can be solved by solving n
 293 different 1-dimensional problems. This reduction allows the resolution of Problem (8) by
 294 Corollary 2.1. In particular, the results in this section show that if the l_1 -norm is used, the
 295 optimal policy is to park (to fix) the service unit at some vertex of the region where the
 296 random incident takes place.
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3. THE TWO-FACILITY PROBLEM

302 In the previous section, we considered the problem of locating only one facility to
 303 cover a random incident. However, often more than one service unit is necessary,
 304 especially if the coverage region is large. In this section, we consider the case where two
 305 service facilities cover a random incident under the usual nearest allocation rule: the
 306 random incident is covered by the closest service unit. This allocation rule leads to the
 307 following formulation:
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$$309$$

$$310 \min_{X_1, X_2 \in \mathcal{F}_1} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]^2} \int_{[0,1]} \min\{|x_1 - a|, |x_2 - a|\} dG_A(a) dF_{1,2}(x_1, x_2), \quad (10)$$

$$311$$

$$312$$

313 where $\mathcal{F}_1, \mathcal{G}_1(\mu)$ were defined in Section 2, $G_A(\cdot)$ is the c.d.f. of the random variable A and
 314 $F_{1,2}(\cdot, \cdot)$ is the joint c.d.f. of the random variables X_1 and X_2 . It is worth noting that this is a
 315 non-trivial problem: 1) it is a minmax problem, and 2) the decision space is a functional
 316 space of random vectors.

317 This formulation allows us to model different real-world situations where there are
 318 two-service units to cover a random incident. This is for example the case of highways
 319 with two patrolling vehicles so that each one covers the closest incident.

320 In order to solve this problem, first we consider the case where the servers are not
 321 allowed to patrol, that is, X_1 and X_2 are degenerate random variables. After that, we deal
 322 with the general case: X_1 and X_2 are any random variables belonging to \mathcal{F}_1 .

The formulation of Problem (10) for the first case is given by the following expression

$$\min_{x_1, x_2 \in [0, 1]} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0, 1]} \min\{|x_1 - a|, |x_2 - a|\} dG_A(a). \quad (11)$$

Remark 3.1 Without loss of generality we can assume that $x_1 \leq x_2$.

Before we proceed to obtain the solution of Problem (11), we define the following functions:

$$\bar{d}(x_1, x_2, A) := 2 \left(\int_0^{x_1} G_A(a) da + \int_{\frac{x_1+x_2}{2}}^{x_2} G_A(a) da \right) + \mu - x_2, \quad (12)$$

$$C(x_1, x_2, p_1) := 2 \left((1 - \mu) \frac{x_1 + x_2}{2} - p_1 \left(x_1 - \left(\frac{x_1 + x_2}{2} \right)^2 \right) \right) - x_2 + \mu, \quad (13)$$

and the set

$$T(x_1, x_2) := \{p = (p_0, p_1, p_2) \geq 0 : p_0 + p_1 + p_2 = 1 \\ \text{and } p_1 \frac{x_1 + x_2}{2} + p_2 = \mu\}, \quad (14)$$

where A is a random variable in $\mathcal{G}_1(\mu)$ with distribution function G_A and $x_1, x_2 \in [0, 1]$.

Theorem 3.1 *The optimal positioning policy in the hypothesis of Problem (11) is $x_1 = \mu^2$ and $x_2 = 2\mu - \mu^2$.*

Proof: First, by Lemma A.5 and A.7, we have that

$$\min_{0 \leq x_1 \leq x_2 \leq 1} \max_{A \in \mathcal{G}_1(\mu)} \int_{[0, 1]} \min\{|x_1 - a|, |x_2 - a|\} dG_A(a) = \min_{0 \leq x_1 \leq x_2 \leq 1} \max_{p \in T(x_1, x_2)} C(x_1, x_2, p_1),$$

where C and $T(x_1, x_2)$ were defined in (13) and (14), respectively. By Lemma A.8 the optimal solution of this problem is $x_1 = \mu^2$ and $x_2 = 2\mu - \mu^2$ and the proof is concluded. \square

Once we have studied the problem of locating two deterministic service units, we consider the general problem where the service units are random vectors. In this case we consider the original Problem (10).

Theorem 3.2 *The optimal positioning policy of Problem (10) are the random variables X_1^* and X_2^* such that $P[X_1^* = \mu^2] = 1$ and $P[X_2^* = 2\mu - \mu^2] = 1$.*

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369 **Proof:** Using Lemma A.5, we can bound the expression in (10) as follows (recall that \bar{d}
370 was defined in (12)):

$$\begin{aligned}
 & \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]^2} \int_{[0,1]} \min\{|x_1 - a|, |x_2 - a|\} dG_A(a) dF_{1,2}(x_1, x_2) \\
 &= \max_{A \in \mathcal{G}_1(\mu)} \left[\int_{[0,x_2] \times [0,1]} \bar{d}(x_1, x_2, A) dF_{1,2}(x_1, x_2) + \int_{(x_2,1] \times [0,1]} \bar{d}(x_2, x_1, A) dF_{1,2}(x_1, x_2) \right] \\
 &\leq \int_{[0,x_2] \times [0,1]} \max_{A \in \mathcal{G}_1(\mu)} \bar{d}(x_1, x_2, A) dF_{1,2}(x_1, x_2) \\
 &\quad + \int_{(x_2,1] \times [0,1]} \max_{A \in \mathcal{G}_1(\mu)} \bar{d}(x_2, x_1, A) dF_{1,2}(x_1, x_2).
 \end{aligned} \tag{15}$$

383 Define

$$384 \quad S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2 \leq 1\},$$

$$385 \quad S_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 < x_1 \leq 1\},$$

386
387 and let $\mathcal{X}_{S_j}(\cdot)$ denote the indicator function of the set S_j for $j = 1, 2$. Now, Lemma A.7
388 allows to write the integrands in (15) as

$$389 \quad \max_{A \in \mathcal{G}_1(\mu)} \bar{d}(x_1, x_2, A) = \max_{p \in T(x_1, x_2)} C(x_1, x_2, p_1) \text{ for } (x_1, x_2) \in S_1, \tag{16}$$

$$390 \quad \max_{A \in \mathcal{G}_1(\mu)} \bar{d}(x_2, x_1, A) = \max_{p \in T(x_1, x_2)} C(x_2, x_1, p_1) \text{ for } (x_1, x_2) \in S_2. \tag{17}$$

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396 Combining (16) and (17) we can rewrite the expression (15) as

$$397 \quad \int_{[0,1]^2} \max_{p \in T(x_1, x_2)} [C(x_1, x_2, p_1) \mathcal{X}_{S_1}(x_1, x_2) + C(x_2, x_1, p_1) \mathcal{X}_{S_2}(x_1, x_2)] dF_{1,2}(x_1, x_2).$$

398
399 Let $p^*(x_1, x_2) = (p_0^*(x_1, x_2), p_1^*(x_1, x_2), p_2^*(x_1, x_2)) \in T(x_1, x_2)$ be the function where the
400 expression above reaches its inner maximum. Notice that the expression of $p^*(x_1, x_2)$ can
401 be obtained from the proof of Lemma A.8, and it is defined in a different way depending on
402 the region that (x_1, x_2) belongs to.

403
404 Now, for all $(x_1, x_2) \in [0, 1]^2$, let $A^*(x_1, x_2)$ be a random variable independent of (X_1, X_2) ,
405 whose probability distribution is $dG_{A^*(x_1, x_2)}$, defined by

$$406 \quad dG_{A^*(x_1, x_2)}(a) = \begin{cases} p_0^*(x_1, x_2) & \text{if } a = 0 \\ p_1^*(x_1, x_2) & \text{if } a = \frac{x_1 + x_2}{2} \\ p_2^*(x_1, x_2) & \text{if } a = 1. \end{cases} \tag{18}$$

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409 Notice that, for a fixed x_1 and x_2 belonging to the interval $[0,1]$, $A^*(x_1, x_2)$ is a discrete
410 random variable taking the values $0, \frac{x_1 + x_2}{2}$ and 1 with probabilities $p_0^*(x_1, x_2), p_1^*(x_1, x_2)$ and
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415 $p_2^*(x_1, x_2)$ respectively. Thus, by the definition of the functions \bar{d} and C (see the proof of
416 Lemma A.7), $A^*(x_1, x_2)$ verifies that

$$417 \quad \bar{d}(x_1, x_2, A^*(x_1, x_2)) = C(x_1, x_2, p_1^*(x_1, x_2)) \text{ with } (x_1, x_2) \in S_1, \quad (19)$$

$$418 \quad \bar{d}(x_2, x_1, A^*(x_1, x_2)) = C(x_2, x_1, p_1^*(x_1, x_2)) \text{ with } (x_1, x_2) \in S_2. \quad (20)$$

421 Now, using (19), (20), and Lemma A.5, we have that

$$\begin{aligned} 422 & \int_{[0,1]^2} \max_{p \in T(x_1, x_2)} [C(x_1, x_2, p_1) \mathcal{X}_{S_1}(x_1, x_2) + C(x_2, x_1, p_1) \mathcal{X}_{S_2}(x_1, x_2)] dF_{1,2}(x_1, x_2) \\ 423 & = \int_{[0,1]^2} [C(x_1, x_2, p_1^*(x_1, x_2)) \mathcal{X}_{S_1}(x_1, x_2) \\ 424 & \quad + C(x_2, x_1, p_1^*(x_1, x_2)) \mathcal{X}_{S_2}(x_1, x_2)] dF_{1,2}(x_1, x_2) \\ 425 & = \int_{[0, x_2] \times [0, 1]} \bar{d}(x_1, x_2, A^*(x_1, x_2)) dF_{1,2}(x_1, x_2) \\ 426 & \quad + \int_{[x_2, 1] \times [0, 1]} \bar{d}(x_2, x_1, A^*(x_1, x_2)) dF_{1,2}(x_1, x_2) \\ 427 & = \int_{[0,1]^2} \int_{[0,1]} \min\{|x_1 - a|, |x_2 - a|\} dG_{A^*(x_1, x_2)}(a) dF_{1,2}(x_1, x_2). \end{aligned}$$

428 Therefore, using the inequality in (15), we have that

$$\begin{aligned} 429 & \max_{A \in \mathcal{G}_1(\mu)} \int_{[0,1]^2} \int_{[0,1]} \min\{|x_1 - a|, |x_2 - a|\} dG_A(a) dF_{1,2}(x_1, x_2) \\ 430 & \leq \int_{[0,1]^2} \int_{[0,1]} \min\{|x_1 - a|, |x_2 - a|\} dG_{A^*(x_1, x_2)}(a) dF_{1,2}(x_1, x_2). \end{aligned}$$

431 Moreover, since $p^*(x_1, x_2) \in T(x_1, x_2)$, we have that the mean value of $A^*(x_1, x_2)$ is μ for
432 all $(x_1, x_2) \in [0, 1]^2$. Therefore, $A^*(X_1, X_2)$ is also a random variable with mean value μ ,
433 that is, $A^*(X_1, X_2) \in \mathcal{G}_1(\mu)$. Thus, the inequality above has to be an equality and Problem
434 (10) can be reformulated as follows:

$$\begin{aligned} 435 & \min_{x_1, x_2 \in \mathcal{F}_1} \int_{[0,1]^2} [C(x_1, x_2, p_1^*(x_1, x_2)) \mathcal{X}_{S_1}(x_1, x_2) \\ 436 & \quad + C(x_2, x_1, p_1^*(x_1, x_2)) \mathcal{X}_{S_2}(x_1, x_2)] dF_{1,2}(x_1, x_2). \end{aligned} \quad (21)$$

437 Let us define the following function;

$$438 \quad L(x_1, x_2) := C(x_1, x_2, p_1^*(x_1, x_2)) \mathcal{X}_{S_1}(x_1, x_2) + C(x_2, x_1, p_1^*(x_1, x_2)) \mathcal{X}_{S_2}(x_1, x_2).$$

461 Since, it always holds that,

$$462 \min_{x_1, x_2 \in \mathcal{F}_1} \int_{[0,1]^2} L(x_1, x_2) dF_{1,2}(x_1, x_2) = \min_{x_1, x_2 \in [0,1]} L(x_1, x_2),$$

463 then the minimum in (21) is reached by two degenerate random variables. On the other
464 hand, using that the function $L(x_1, x_2)$ is defined in disjoint sets,
465 $C(x_1, x_2, p_1^*(x_1, x_2))\mathcal{A}_{S_1}(x_1, x_2) \geq 0$ and $C(x_2, x_1, p_1^*(x_1, x_2))\mathcal{A}_{S_2}(x_1, x_2) \geq 0$ (see (19), (20)
466 and Lemma A.5 to justify the non-negativity of these functions), we have that

$$467 L(x_1, x_2) = \max \{ C(x_1, x_2, p_1^*(x_1, x_2))\mathcal{A}_{S_1}(x_1, x_2), C(x_2, x_1, p_1^*(x_1, x_2))\mathcal{A}_{S_2}(x_1, x_2) \}.$$

472 Therefore, we can use the same arguments as in the deterministic case (Lemma A.8) in
473 order to obtain that the optimal solutions are the random vectors (X_1, X_2) such that:

$$474 P[(X_1, X_2) = (\mu^2, 2\mu - \mu^2)] = 1 \quad \text{or} \quad P[(X_1, X_2) = (2\mu - \mu^2, \mu^2)] = 1. \quad \square$$

475 In conclusion, Theorem 3.2 proves that it is optimal to park the service units when no
476 hypotheses are made on their c.d.f.'s and only the mean value of the position of the random
477 incident is known.

483 4. CONCLUDING REMARKS

484 The results in this paper extend other previously known results about the optimal
485 location of one or two service units to situations where no assumptions are made on the
486 probability distribution of the random incident that these service units cover apart from
487 the knowledge of its mean value (whereas all the previous papers require exact knowledge
488 of this distribution). This is accomplished by minimizing the maximum expected response
489 time (whereas the previous results minimize expected distances). In particular, we show
490 that when the only available information is the mean value of the position of the incidents,
491 then the optimal policy is to park the service units at concrete points.

492 On the other hand, since our goal is to minimize the response time from the service
493 unit to the incident, another interesting problem is to assume the same probability
494 distribution for both the service unit and the incident. Notice that one interpretation of this
495 policy is that we are fixing the location of the service unit at the location of the previous
496 incident. The worst case for this policy is given by

$$497 \max_{F \in \mathcal{F}_1(\mu)} \int_{[0,1]} \int_{[0,1]} \|x - a\|_1 dF(a) dF(x)$$

500 (assuming that incidents occur independently). Using a similar argument to the one used in
501 the proof of Theorem 2.2, we have that

$$502 \max_{F \in \mathcal{F}_1(\mu)} \int_{[0,1]} \int_{[0,1]} \|x - a\|_1 dF(a) dF(x) = \mu + \mu(1 - 2\mu).$$

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507 Thus, the maximum objective value above is $1/2$ and it is achieved by a c.d.f. F with mean
 508 value $\mu = 1/2$. Moreover, by the proof of Corollary 2.1, we have that the objective value
 509 of the worst case for Problem (6) is $\mu + \lambda(1 - 2\mu)$. Thus, the best objective value when
 510 λ varies, is:

$$511 \quad \begin{cases} \mu & \text{if } \mu \leq 0.5 \text{ and it is achieved at } \lambda^* = 0 \\ 512 \\ 513 \quad 1 - \mu & \text{if } \mu > 0.5 \text{ and it is achieved at } \lambda^* = 1. \end{cases}$$

514 Therefore, for a fixed μ , the difference in worst case performance between the approach
 515 considered above and the approach of the paper is given by

$$516 \quad |(\lambda^* - \mu)(1 - 2\mu)| = \begin{cases} \mu(1 - 2\mu) & \text{if } \mu \leq 0.5 \\ 517 \\ 518 \quad (1 - \mu)(-1 + 2\mu) & \text{if } \mu > 0.5. \end{cases}$$

519 Notice that there is no difference when $\mu \in \{0, 0.5, 1\}$, but that there is a difference that
 520 can be quantified for other choices of μ . We can see that this difference is maximal when
 521 $\mu = 0.25$ and $\mu = 0.75$ which is reasonable because these points are at the maximum
 522 distances to the values of μ where the difference is null.

523 Finally, we can also study the natural extension of Problem (10) where we consider
 524 k service units instead of two. It should be noted that using similar arguments to those
 525 used for the case $k = 2$, we can obtain that the worst case in the distribution of the
 526 incident is given by a random variable taking the values $0, \frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}, \dots, \frac{x_{k-1}+x_k}{2}, 1$ (see
 527 Eq. (18) for the case when $k = 2$). However, the complexity of the expressions obtained
 528 in the analysis does not allow us to present an explicit formula of the optimal solution
 529 of this problem.

530 APPENDIX

531 In this section, we include for the sake of completeness, several results and their
 532 proofs which have been used in the paper.

533 **Lemma A.1** *Let $G(\cdot)$ be a nondecreasing function such that $G: \mathbb{R} \rightarrow [0, \infty)$. If there exists
 534 $M \in \mathbb{R}$ such that*

$$535 \quad \int_a^b G(t) dt = M(b - a) \quad \text{with } a, b \in \mathbb{R}$$

536 then

$$537 \quad I_G(z) := \int_a^z G(t) dt - M(z - a) \leq 0 \quad \forall z \in [a, b].$$

538 **Proof:** Let $t_0 \in [a, b]$ be such that $t_0 = \inf \{t : G(t) \geq M\}$;

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- 553 i) If $z < t_0$ we have that $G(t) < M \quad \forall t \leq z$ thus, $I_G(z) \leq 0$.
 554 ii) If $z \geq t_0$ we obtain that,

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where we have used the fact that the function $G(\cdot)$ is nondecreasing. Thus, the lemma is proved. \square

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Lemma A.2 For any $A \in \mathcal{G}_1(\mu)$ with c.d.f. $G_A(\cdot)$, we have that:

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Proof: Denote D_G the set of denumerable number of discontinuity points of $G_A(\cdot)$ in the interval $[0,1]$ union with the set $\{0,1\}$. Applying integration by parts to the interval (x_{i-1}, x_i) where x_{i-1} and x_i are two consecutive points of D_G , we have that

$$\begin{aligned}
 \int_{(x_{i-1}, x_i)} G_A(a) da &= aG_A(a) \Big|_{x_{i-1}^+}^{x_i^-} - \int_{(x_{i-1}, x_i)} a dG_A(a) \\
 &= x_i^- G_A(x_i^-) - x_{i-1}^+ G_A(x_{i-1}^+) - \int_{(x_{i-1}, x_i)} a dG_A(a) \\
 &= x_i(G_A(x_i) - P[A = x_i]) - x_{i-1}G_A(x_{i-1}) - \int_{(x_{i-1}, x_i)} a dG_A(a) \\
 &= x_i G_A(x_i) - x_{i-1} G_A(x_{i-1}) - \int_{(x_{i-1}, x_i)} a dG_A(a).
 \end{aligned}$$

If we sum the equality above for each element of D_G we obtain

$$\begin{aligned}
 \sum_{x_i \in D_G} \int_{(x_{i-1}, x_i)} G_A(a) da &= \sum_{x_i \in D_G} \left[x_i G_A(x_i) - x_{i-1} G_A(x_{i-1}) - \int_{(x_{i-1}, x_i)} a dG_A(a) \right] \\
 &= 1 - \int_{(0,1)} a dG_A(a).
 \end{aligned}$$

Hence, since the integrand in the last expression is null at zero we have that

$$\int_0^1 G_A(a) da = 1 - \int_{[0,1]} a dG_A(a) = 1 - \mu.$$

Now we prove the second assertion. We have the following equalities

$$\begin{aligned} \int_{[0,1]} |x - a| dG_A(a) &= \int_{[0,x]} (x - a) dG_A(a) + \int_{(x,1]} (a - x) dG_A(a) \\ &= xG_A(x) - \int_{[0,x]} a dG_A(a) + \int_{(x,1]} a dG_A(a) - x(1 - G_A(x)) \\ &= x(2G_A(x) - 1) - \int_{[0,x]} a dG_A(a) + \mu - \int_{[0,x]} a dG_A(a) \\ &= 2(xG_A(x) - \int_{[0,x]} a dG_A(a)) + \mu - x. \end{aligned}$$

Applying integration by parts using the arguments above we have that

$$\int_0^x G_A(a) da = xG_A(x) - \int_{[0,x]} a dG_A(a),$$

and the result follows. \square

Lemma A.3 For any $X \in \mathcal{F}_1(\lambda)$ and $A \in \mathcal{G}_1(\mu)$ with c.d.f's $F_X(\cdot)$ and $G_A(\cdot)$, respectively, we have that

$$\begin{aligned} \int_{[0,1]} yG_A(y) dF_X(y) + \int_{[0,1]} yF_X(y) dG_A(y) &= 1 + \sum_{y \in D} yP[X = y]P[A = y] \\ &\quad - \int_0^1 G_A(y)F_X(y) dy, \end{aligned}$$

where D is the set of denumerable number of discontinuity points either of $F_X(\cdot)$ or $G_A(\cdot)$ (or both) union with the set $\{0,1\}$.

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Proof: Applying integration by parts to the interval (x_{i-1}, x_i) where x_{i-1} and x_i are two consecutive points of D , we have the following equalities

$$\begin{aligned}
 & \int_{(x_{i-1}, x_i)} y G_A(y) dF_X(y) + \int_{(x_{i-1}, x_i)} y F_X(y) dG_A(y) \\
 &= y G_A(y) F_X(y) \Big|_{x_{i-1}^+}^{x_i^-} - \int_{(x_{i-1}, x_i)} G_A(y) F_X(y) dy \\
 &= x_i^- G_A(x_i^-) F_X(x_i^-) - x_{i-1}^+ G_A(x_{i-1}^+) F_X(x_{i-1}^+) - \int_{x_{i-1}}^{x_i} G_A(y) F_X(y) dy \\
 &= x_i (G_A(x_i) - P[A = x_i]) (F_X(x_i) - P[X = x_i]) - x_{i-1} G_A(x_{i-1}) F_X(x_{i-1}) \\
 &\quad - \int_{x_{i-1}}^{x_i} G_A(y) F_X(y) dy = x_i G_A(x_i) F_X(x_i) - x_{i-1} G_A(x_{i-1}) F_X(x_{i-1}) \\
 &\quad - x_i P[X = x_i] G_A(x_i) - x_i P[A = x_i] F_X(x_i) + x_i P[A = x_i] P[X = x_i] \\
 &\quad - \int_{x_{i-1}}^{x_i} G_A(y) F_X(y) dy.
 \end{aligned}$$

The equality above can be rewritten as

$$\begin{aligned}
 & \int_{(x_{i-1}, x_i]} y G_A(y) dF_X(y) + \int_{(x_{i-1}, x_i]} y F_X(y) dG_A(y) \\
 &= x_i G_A(x_i) F_X(x_i) - x_{i-1} G_A(x_{i-1}) F_X(x_{i-1}) \\
 &\quad + x_i P[A = x_i] P[X = x_i] - \int_{x_{i-1}}^{x_i} G_A(y) F_X(y) dy.
 \end{aligned}$$

If we sum the expression above for each element of D we have

$$\begin{aligned}
 & \sum_{x_i \in D} \left[\int_{(x_{i-1}, x_i]} y G_A(y) dF_X(y) + \int_{(x_{i-1}, x_i]} y F_X(y) dG_A(y) \right] \\
 &= \sum_{x_i \in D} \left[x_i G_A(x_i) F_X(x_i) - x_{i-1} G_A(x_{i-1}) F_X(x_{i-1}) - + x_i P[A = x_i] P[X = x_i] \right. \\
 &\quad \left. - \int_{x_{i-1}}^{x_i} G_A(y) F_X(y) dy \right] = 1 + \sum_{x_i \in D} x_i P[A = x_i] P[X = x_i] - \int_0^1 G_A(y) F_X(y) dy.
 \end{aligned}$$

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Since, at zero the integrands of the first part of the equalities above are null, we have that

$$\int_{[0,1]} yG_A(y) dF_X(y) + \int_{[0,1]} yF_X(y) dG_A(y) = 1 + \sum_{x_i \in D} x_i P[A = x_i] P[X = x_i] - \int_0^1 G_A(y) F_X(y) dy,$$

and the result follows. \square

Lemma A.4 For any $X \in \mathcal{F}_1(\lambda)$ and $A \in \mathcal{G}_1(\mu)$ with c.d.f's $F_X(\cdot)$ and $G_A(\cdot)$, respectively, we have that

$$\int_{[0,1]} \int_{[0,1]} |x-a| dG_A(a) dF_X(x) = 2 \left(1 - \int_0^1 G_A(y) F_X(y) dy \right) - \lambda - \mu.$$

Proof: We have that

$$\begin{aligned} & \int_{[0,1]} \int_{[0,1]} |x-a| dG_A(a) dF_X(x) \\ &= \int_{[0,1]} \int_{[0,x]} x dG_A(a) dF_X(x) - \int_{[0,1]} \int_{[0,x]} a dG_A(a) dF_X(x) \\ & \quad + \int_{[0,1]} \int_{(x,1]} a dG_A(a) dF_X(x) - \int_{[0,1]} \int_{(x,1]} x dG_A(a) dF_X(x) \\ &= \int_{[0,1]} xG_A(x) dF_X(x) - \int_{[0,1]} \int_{[0,x]} a dG_A(a) dF_X(x) \\ & \quad + \int_{[0,1]} \int_{(x,1]} a dG_A(a) dF_X(x) - \int_{[0,1]} x(1-G_A(x)) dF_X(x) \\ &= 2 \int_{[0,1]} xG_A(x) dF_X(x) - \lambda - \int_{[0,1]} \int_{[0,x]} a dG_A(a) dF_X(x) \\ & \quad + \int_{[0,1]} (\mu - \int_{[0,x]} a dG_A(a) dF_X(x)) dF_X(x) = 2 \int_{[0,1]} xG_A(x) dF_X(x) \\ & \quad - 2 \int_{[0,1]} \int_{[0,x]} a dG_A(a) dF_X(x) - \lambda + \mu = 2 \int_{[0,1]} xG_A(x) dF_X(x) \\ & \quad - 2 \int_{[0,1]} a \int_{[a,1]} dF_X(x) dG_A(a) - \lambda + \mu = 2 \int_{[0,1]} xG_A(x) dF_X(x) \\ & \quad - 2 \int_{[0,1]} a \int_{(a,1]} dF_X(x) dG_A(a) - 2 \int_{[0,1]} aP[X = a] dG_A(a) - \lambda + \mu. \end{aligned}$$

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737 Denote by D the denumerable number of discontinuity points of F_X or G_A union with
 738 $\{0,1\}$. Then, we rewrite the expression above as

$$\begin{aligned}
 739 & 2 \int_{[0,1]} x G_A(x) dF_X(x) - 2\mu + 2 \int_{[0,1]} a F_X(a) dG_A(a) - 2 \sum_{y \in D} y P[X=y] P[A=y] - \lambda + \mu \\
 740 & \\
 741 & \\
 742 & \\
 743 & = 2 \left(\int_{[0,1]} y G_A(y) dF_X(y) + \int_{[0,1]} y F_X(y) dG_A(y) \right) - 2 \sum_{y \in D} y P[X=y] P[A=y] - \lambda - \mu. \\
 744 & \\
 745 & \hspace{15em} (22) \\
 746 &
 \end{aligned}$$

747 Now, using Lemma A.3 we have that

$$\begin{aligned}
 748 & \\
 749 & (22) = 2 \left(1 - \int_0^1 G_A(y) F_X(y) dy \right) - \lambda - \mu, \\
 750 & \\
 751 &
 \end{aligned}$$

752 and the result follows. \square

753 **Lemma A.5** For any $A \in \mathcal{G}_1(\mu)$ with c.d.f. $G_A(\cdot)$, the function \bar{d} defined in (12) admits
 754 the following representation.

$$\begin{aligned}
 755 & \\
 756 & \\
 757 & \bar{d}(x_1, x_2, A) = \int_{[0,1]} \min\{|x_1 - a|, |x_2 - a|\} dG_A(a) \quad \forall x_1 \leq x_2 \in [0, 1]. \\
 758 & \\
 759 &
 \end{aligned}$$

760 **Proof:** We have the following equalities:

$$\begin{aligned}
 761 & \\
 762 & \\
 763 & \\
 764 & \int_{[0,1]} \min\{|x_1 - a|, |x_2 - a|\} dG_A(a) = \int_{[0, \frac{x_1+x_2}{2}]} |x_1 - a| dG_A(a) + \int_{(\frac{x_1+x_2}{2}, 1]} |x_2 - a| dG_A(a) \\
 765 & \\
 766 & \\
 767 & = \int_{[0, x_1]} (x_1 - a) dG_A(a) + \int_{(x_1, \frac{x_1+x_2}{2}]} (a - x_1) dG_A(a) \\
 768 & \\
 769 & \\
 770 & + \int_{(\frac{x_1+x_2}{2}, x_2]} (x_2 - a) dG_A(a) + \int_{(x_2, 1]} (a - x_2) dG_A(a) \\
 771 & \\
 772 & \\
 773 & = 2x_1 G_A(x_1) - G_A\left(\frac{x_1+x_2}{2}\right)(x_1+x_2) + 2x_2 G_A(x_2) \\
 774 & \\
 775 & \\
 776 & - x_2 - \int_{[0, x_1]} a dG_A(a) + \int_{(x_1, \frac{x_1+x_2}{2}]} a dG_A(a) \\
 777 & \\
 778 & \\
 779 & - \int_{(\frac{x_1+x_2}{2}, x_2]} a dG_A(a) + \int_{(x_2, 1]} a dG_A(a) \\
 780 & \\
 781 & \\
 782 & = 2(x_1 G_A(x_1) - \int_{[0, x_1]} a dG_A(a) + x_2 G_A(x_2)
 \end{aligned}$$

$$\begin{aligned}
& -G_A\left(\frac{x_1+x_2}{2}\right)\frac{x_1+x_2}{2} - \int_{\left(\frac{x_1+x_2}{2}, x_2\right]} a dG_A(a) \\
& + \mu - x_2 = 2\left(\int_0^{x_1} G_A(a) da + \int_{\frac{x_1+x_2}{2}}^{x_2} G_A(a) da\right) \\
& + \mu - x_2 = \bar{d}(x_1, x_2, A),
\end{aligned}$$

and the result is proved. \square

Lemma A.6 For each random variable $A \in \mathcal{G}_1(\mu)$ with distribution function $G_A(\cdot)$ and each $0 \leq x_1 \leq x_2 \leq 1$, there exists a discrete random variable $\bar{A} \in \mathcal{G}_1(\mu)$ defined by

$$P[\bar{A} = a] = \begin{cases} p_0 & \text{if } a = 0 \\ p_1 & \text{if } a = \frac{x_1+x_2}{2} \\ p_2 & \text{if } a = 1, \end{cases}$$

where (p_0, p_1, p_2) satisfies that

$$p_0 + p_1 + p_2 = 1$$

$$\bar{d}(x_1, x_2, A) \leq \bar{d}(x_1, x_2, \bar{A})$$

$$\int_0^{\frac{x_1+x_2}{2}} G_A(a) da = p_0 \frac{x_1+x_2}{2} \tag{23}$$

$$\int_{\frac{x_1+x_2}{2}}^1 G_A(a) da = (p_0 + p_1) \left(1 - \frac{x_1+x_2}{2}\right). \tag{24}$$

Remark A.1 It should be noted that from this result and part i) of Lemma A.2, one obtains that for each $0 \leq x_1 \leq x_2 \leq 1$, the $\max_{A \in \mathcal{G}_1(\mu)} \bar{d}(x_1, x_2, A)$ is attained in a discrete random variable with mean value μ and defined only on the values $0, \frac{x_1+x_2}{2}$ and 1 . Moreover, $p = (p_0, p_1, p_2) \in T(x_1, x_2)$ (where T was defined in (14)).

Proof: First, we note that $\bar{A} \in \mathcal{G}_1(\mu)$, see Remark A.1.

Second, in order to complete the proof of the lemma it suffices to prove:

- i) $\int_0^{x_1} G_A(a) da - p_0 x_1 \leq 0$
- ii) $\int_{\frac{x_1+x_2}{2}}^{x_2} G_A(a) da - (p_0 + p_1) \left(x_2 - \frac{x_1+x_2}{2}\right) \leq 0$.

To this end, we apply Lemma A.1. Since (23) and (24) hold and $G_A(\cdot)$ is a probability distribution function, we are under hypotheses of Lemma A.1 and thus, i) and ii) are proved. \square

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829 **Lemma A.7** *If $x_1, x_2 \in [0, 1]$, $x_1 \leq x_2$, then*

$$830 \quad \max_{A \in \mathcal{G}_1(\mu)} \bar{d}(x_1, x_2, A) = \max_{p \in T(x_1, x_2)} C(x_1, x_2, p_1),$$

832

833 *where \bar{d} , C , and T were defined in (12), (13), and (14), respectively.*

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837 **Proof:** We have, by the definition of \bar{d} in (12) and Lemma A.6, that;

$$838 \quad \max_{A \in \mathcal{G}_1(\mu)} \bar{d}(x_1, x_2, A) = \max_{A \in \mathcal{G}_1(\mu)} 2 \left(\int_0^{x_1} G_A(a) da + \int_{\frac{x_1+x_2}{2}}^{x_2} G_A(a) da \right) + \mu - x_2$$

$$840 \quad = \max_{p \in T(x_1, x_2)} 2 \left(p_0 x_1 + (p_0 + p_1) \left(x_2 - \frac{x_1 + x_2}{2} \right) \right) + \mu - x_2.$$

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845 Using that $p \in T(x_1, x_2)$, (i.e., $p_0 + p_1 + p_2 = 1$ and $\frac{x_1+x_2}{2} p_1 + p_2 = \mu$) we have that

846

$$847 \quad \max_{p \in T(x_1, x_2)} 2 \left(p_0 x_1 + (p_0 + p_1) \left(x_2 - \frac{x_1 + x_2}{2} \right) \right) + \mu - x_2$$

848

849

$$850 \quad = \max_{p \in T(x_1, x_2)} \mu - x_2 + 2 \left((1 - \mu + p_1 \left(\frac{x_1 + x_2}{2} - 1 \right)) x_1 \right.$$

851

852

$$853 \quad \left. + (1 - \mu + p_1 \frac{x_1 + x_2}{2}) (x_2 - \frac{x_1 + x_2}{2}) \right)$$

854

855

$$856 \quad = \max_{p \in T(x_1, x_2)} \mu - x_2 + 2 \left((1 - \mu) \frac{x_1 + x_2}{2} - p_1 \left(x_1 - \left(\frac{x_1 + x_2}{2} \right)^2 \right) \right)$$

857

858

$$859 \quad = \max_{p \in T(x_1, x_2)} C(x_1, x_2, p_1),$$

860

861

862 which proves the result. □

863

864

865 **Lemma A.8** *The optimal solution for the problem*

866

$$867 \quad \min_{0 \leq x_1 \leq x_2 \leq 1} \max_{p \in T(x_1, x_2)} C(x_1, x_2, p_1) \quad (25)$$

868

869 *is $x_1 = \mu^2$ and $x_2 = 2\mu - \mu^2$, where C and T were defined in (13) and (14), respectively.*

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Proof: Since $C(x_1, x_2, p_1)$ is linear with respect to p_1 , we analyze the cases where the coefficient that multiplies p_1 is positive or negative separately.

875 **Case 1:** $x_1 \geq \left(\frac{x_1+x_2}{2}\right)^2$

876 In this case, the function $C(x_1, x_2, p_1)$ is decreasing in p_1 , thus the maximum is reached at
877 $p_1 = 0$. That means that $p_0 = 1 - \mu$ and $p_2 = \mu$. Therefore, the expression that we have to
878 consider is the following:
879

$$880 \min_{0 \leq x_1 \leq x_2 \leq 1} C_1(x_1, x_2) := 2 \left((1 - \mu) \frac{x_1 + x_2}{2} \right) - x_2 + \mu \quad (26)$$

$$883 \text{ s.t. } x_1 \geq \left(\frac{x_1 + x_2}{2} \right)^2.$$

886 It is clear that:

$$888 \frac{\partial C_1(x_1, x_2)}{\partial x_1} = 1 - \mu \geq 0.$$

889 Since the function $C_1(x_1, x_2)$ is increasing in x_1 and $x_1 \geq \left(\frac{x_1+x_2}{2}\right)^2$ we have that

$$891 C_1(x_1, x_2) \geq C_1(t, x_2),$$

892 where $t = \left(\frac{t+x_2}{2}\right)^2$. Thus, since $x_2 \geq 0$, this implies that $x_2 = 2\sqrt{t} - t$ (notice that $2\sqrt{t} - t \geq t$ for all $t \geq 0$). Therefore, solving (26) is equivalent to solving the following problem;

$$897 \min_{x_1 \in [0,1]} 2(1 - \mu)\sqrt{x_1} - 2\sqrt{x_1} + x_1 + \mu = \min_{x_1 \in [0,1]} x_1 + (1 - 2\sqrt{x_1})\mu,$$

898 and this problem reaches its minimum at the point $x_1 = \mu^2$. Hence, $x_2 = 2\mu - \mu^2$ and the
899 minimum objective value is $\mu - \mu^2$. \square
900
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903

904 **Case 2:** $x_1 \leq \left(\frac{x_1+x_2}{2}\right)^2$

905 Notice that in Case 1, we have already studied the points (0,0) and (1,1). Therefore, in what
906 follows, we can assume without loss of generality that (x_1, x_2) is neither (0,0) nor (1,1).
907 In this case, the function $C(x_1, x_2, p_1)$ is increasing in p_1 . Since $p \in T(x_1, x_2)$ we have that
908 $p_0 = 1 - \mu + p_1 \left(\frac{x_1+x_2}{2} - 1\right)$, $p_2 = \mu - p_1 \frac{x_1+x_2}{2}$, $0 \leq p_0 \leq 1$ and $0 \leq p_2 \leq 1$ then, we
909 have that,

$$911 \text{ a) } 0 \leq (1 - \mu) - p_1 \left(1 - \frac{x_1+x_2}{2}\right) \leq 1, \text{ that is, } -\frac{\mu}{1 - \frac{x_1+x_2}{2}} \leq p_1 \leq \frac{1-\mu}{1 - \frac{x_1+x_2}{2}} \text{ if } (x_1, x_2) \neq (1, 1).$$

$$913 \text{ b) } 0 \leq \mu - p_1 \frac{x_1+x_2}{2} \leq 1, \text{ that is, } -\frac{1-\mu}{\frac{x_1+x_2}{2}} \leq p_1 \leq \frac{\mu}{\frac{x_1+x_2}{2}} \text{ if } (x_1, x_2) \neq (0, 0).$$

914 Using that $p_1 \geq 0$, $-\frac{\mu}{1 - \frac{x_1+x_2}{2}} \leq 0$ and $-\frac{1-\mu}{\frac{x_1+x_2}{2}} \leq 0$ the previous conditions reduce to;

$$916 \text{ a) } p_1 \leq \frac{1-\mu}{1 - \frac{x_1+x_2}{2}}$$

$$918 \text{ b) } p_1 \leq \frac{\mu}{\frac{x_1+x_2}{2}}.$$

919 Hence, $p_1 \leq \min \left\{ \frac{1-\mu}{1 - \frac{x_1+x_2}{2}}, \frac{\mu}{\frac{x_1+x_2}{2}} \right\}$ and to study this minimum we distinguish two cases;
920

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145

- 921 • Case 2.1: If $\frac{x_1+x_2}{2} \geq \mu$ then $\frac{1-\mu}{1-\frac{x_1+x_2}{2}} \geq \frac{\mu}{\frac{x_1+x_2}{2}}$, thus $p_1 \leq \frac{\mu}{\frac{x_1+x_2}{2}}$.
- 922 • Case 2.2: If $\frac{x_1+x_2}{2} \leq \mu$ then $\frac{1-\mu}{1-\frac{x_1+x_2}{2}} \leq \frac{\mu}{\frac{x_1+x_2}{2}}$, thus, $p_1 \leq \frac{1-\mu}{1-\frac{x_1+x_2}{2}}$.
- 923

924 Since the function $C(x_1, x_2, p_1)$ is increasing in p_1 , in Case 2.1. its maximum in p_1 is reached
 925 at $p_1 = \frac{\mu}{\frac{x_1+x_2}{2}}$ and in Case 2.2 at $p_1 = \frac{1-\mu}{1-\frac{x_1+x_2}{2}}$.

926 Hence, to find the maximum of the function $C(x_1, x_2, p_1)$ we have the following two cases:

927

- 928 • Case 2.1: $p_1 = \frac{\mu}{\frac{x_1+x_2}{2}}$.
- 929
- 930 • Case 2.2: $p_1 = \frac{1-\mu}{1-\frac{x_1+x_2}{2}}$.
- 931
- 932
- 933

934 **Case 2.1:** $p_1 = \frac{\mu}{\frac{x_1+x_2}{2}}$

935 In this case, Problem (25) reduces to the following optimization problem

936

937
$$\min_{0 \leq x_1 \leq x_2 \leq 1} C_2(x_1, x_2) := x_1 \left(1 - \frac{2\mu}{\frac{x_1+x_2}{2}} \right) + \mu$$

938

939

940

941
$$s.t. : x_1 \leq \left(\frac{x_1+x_2}{2} \right)^2 \mu \leq \frac{x_1+x_2}{2}.$$

942

943

944 We obtain that

945

946
$$\frac{\partial C_2(x_1, x_2)}{\partial x_2} = \mu \frac{x_1}{\left(\frac{x_1+x_2}{2} \right)^2} \geq 0.$$

947

948

949 Therefore, $C_2(x_1, x_2)$ is a increasing function in x_2 . Since in this case, (x_1, x_2) satisfies that
 950 $x_2 \geq 2\mu - x_1$ and $x_2 \geq 2\sqrt{x_1} - x_1$ we have that

951

952
$$C_2(x_1, x_2) \geq C_2(x_1, t)$$

953

954 where

955

- 956 • $t = 2\mu - x_1$ if $x_1 \leq \mu^2$
- 957 • $t = 2\sqrt{x_1} - x_1$ if $x_1 \geq \mu^2$.

958 Thus,

959

a) If $x_1 \leq \mu^2$ we have that

960

961
$$\min_{x_1, x_2 \in [0,1]} C_2(x_1, x_2) = \min_{x_1 \in [0,1]} \mu - x_1.$$

962

b) If $x_1 \geq \mu^2$ we have that

963

964

965
$$\min_{x_1, x_2 \in [0,1]} C_2(x_1, x_2) = \min_{x_1 \in [0,1]} x_1 \left(1 - \frac{2\mu}{\sqrt{x_1}} \right) + \mu.$$

966

967 Both cases give us the same optimal solution $x_1 = \mu^2$ and $x_2 = 2\mu - \mu^2$ and its objective
 968 value is $\mu - \mu^2$.

969
 970

971 **Case 2.2:** $p_1 = \frac{1-\mu}{1-\frac{x_1+x_2}{2}}$
 972

973 In this case, Problem (25) reduces to the following optimization problem
 974

$$975 \min_{0 \leq x_1 \leq x_2 \leq 1} C_3(x_1, x_2) := \min_{x_1, x_2 \in [0,1]} x_2 \left(\frac{1-\mu}{1-\frac{x_1+x_2}{2}} - 1 \right) - x_1 \frac{1-\mu}{1-\frac{x_1+x_2}{2}} + \mu$$

976
 977

$$978 \text{ s.t. } x_1 \leq \left(\frac{x_1 + x_2}{2} \right)^2$$

979

$$980 \mu \geq \frac{x_1 + x_2}{2}.$$

981

982 We obtain that,
 983

$$984 \frac{\partial C_3(x_1, x_2)}{\partial x_1} = \frac{(1-\mu)(x_2-1)}{\left(1-\frac{x_1+x_2}{2}\right)^2} \leq 0.$$

985

986 That means that $C_3(x_1, x_2)$ is a decreasing function in x_1 . Since, in this case, (x_1, x_2) satisfies
 987 that $x_1 \leq \left(\frac{x_1+x_2}{2}\right)^2$ and $\mu \geq \frac{x_1+x_2}{2}$, we have that $x_1 \leq \mu^2$ then
 988

$$989 C_3(x_1, x_2) \geq C_3(\mu^2, x_2).$$

990

991 Thus, taking $x_1 = \mu^2$ we have that $\mu \geq \frac{\mu^2+x_2}{2}$ and $\mu^2 \leq \left(\frac{\mu^2+x_2}{2}\right)^2$, that is, $x_2 \leq 2\mu - \mu^2$
 992 and $x_2 \geq 2\mu - \mu^2$. (Notice that we do not have to consider the other solution $x_2 \geq$
 993 $-2\mu - \mu^2$ since $-2\mu - \mu^2 \leq 0$ and $x_2 \geq 0$). Therefore, $x_2 = 2\mu - \mu^2$.
 994

995 Since, all the cases give us the same optimal solution, the optimal solution to Problem (25)
 996 is $x_1 = \mu^2$ and $x_2 = 2\mu - \mu^2$.
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 998
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